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Resolutions of the identity in terms of line integrals of *SU*(1, 1) coherent states

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Abstract. Resolutions of the identity in terms of contour integrals that involve $SU(1,1)$ coherent states and their 'complementary states', are derived. The complementary states are auxiliary states that help in the formulation of these resolutions of the identity. Since the $SU(1,1)$ coherent states are normalizable inside the unit disc and the complementary states are normalizable outside the unit disc, enlargements of the Hilbert space are considered which allow the construction of resolutions of the identity in terms of contour integrals in rings $1-\epsilon_2 < |z| < 1+\epsilon_1$. Several examples of our formalism are presented.

1. Introduction

Coherent states were first introduced for the Heisenberg–Weyl group and play an important role in many branches of physics [1]. A modern definition of coherent states is to consider displaced vacuum states

$$|\alpha\rangle \equiv D(\alpha,\alpha^*)|0\rangle \qquad D(\alpha,\alpha^*) \equiv \exp\bigl(\alpha a^\dagger - \alpha^* a\bigr) \qquad [a, a^\dagger] = I \tag{1}$$

where $D(\alpha,\alpha^*)$ is the displacement operator. This definition can easily be generalized to other Lie groups, for example the $SU(2)$ and $SU(1,1)$ groups. $SU(1,1)$ coherent states in particular which are of interest to us here, have been studied extensively both from a mathematical (e.g., [2–7]), but also from a more applied point of view in connection with parametric amplifiers [8].

An important property of coherent states is the resolution of the identity which for the coherent states of the Heisenberg–Weyl group is

$$\int \mathrm{d}\mu(\alpha)\ |\alpha\rangle\,\langle\alpha| = I \qquad \mathrm{d}\mu(\alpha) = \frac{\mathrm{d}^2\alpha}{\pi}. \tag{2}$$

It is known that the full set of coherent states (associated with any group) is highly overcomplete, in the sense that there are much smaller subsets which are also overcomplete. In order to exploit these smaller subsets practically we need to find resolutions of the identity in terms of them. In practice this is not easy, and then even weaker structures, like for example the concept of frames, are desirable.

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In this spirit, resolutions of the identity in terms of coherent states on a line were studied in [9, 10]; and related analytic representations were studied in [11]. This work was extended to $SU(2)$ coherent states in [12]. In this paper we present the analogue of this work for $SU(1,1)$ coherent states, i.e. we give resolutions of the identity in terms of line integrals of $SU(1,1)$ coherent states. We stress that this generalization from $SU(2)$ to $SU(1,1)$ is highly non-trivial because of convergence difficulties in the latter case. We introduce resolutions of the identity which can be used not in the full Hilbert space but in smaller spaces which we explicitly describe. In this sense our resolutions of the identity are weaker mathematical structures than the standard resolutions of the identity (which are valid in the full Hilbert space without convergence difficulties).

In section 2 we introduce the complementary states which are auxiliary states for the formulation of resolutions of the identity in terms of line integrals of $SU(1,1)$ coherent states. We show that, apart from being useful for this particular purpose, the complementary states are also interesting in their own right. For example, they form an overcomplete basis and for $k < \frac{1}{2}$ (where k is a parameter defined in section 2 which characterizes the representation) there exists a resolution of the identity in terms of surface integrals of the complementary states.

In section 3 we discuss our central point: resolutions of the identity in terms of contour integrals of $SU(1,1)$ coherent states. We define the spaces in which these resolutions of the identity can be used carefully. In section 4 we show how an arbitrary state can be expanded in terms of $SU(1,1)$ coherent states on a contour, and give several examples. In section 5 we apply these ideas in the context of squeezed states in quantum optics.

In section 6 we extend the resolutions of the identity into 'forbidden regions' of the parameters, by regularizing the relevant divergent integrals. We conclude the paper in section 7 with a discussion of our results.

2. *SU*(1, 1) coherent states and their complementary states

Let K_0, K_+, K_- be the generators of the $SU(1,1)$ group satisfying the commutator relations

$$\begin{aligned} &[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0 \\ &C \equiv K^2 = K_0^2 - \tfrac{1}{2}\big(K_+K_- + K_-K_+\big) \end{aligned} \tag{3}$$

where $C \equiv K^2$ is the Casimir operator. The standard basis for the coadjoint representation $|k,n\rangle$, is defined by the relations

$$\begin{aligned} &K^2|k,n\rangle = k(k-1)|k,n\rangle \\ &K_0|k,n\rangle = (k+n)|k,n\rangle \\ &K_-|k,n\rangle = \big[n(n+2k-1)\big]^{1/2}|k,n-1\rangle \\ &K_+|k,n\rangle = \big[(n+1)(n+2k)\big]^{1/2}|k,n+1\rangle \qquad (n = 0,1,2,\ldots) \end{aligned} \tag{4}$$

where k is a real number characterizing the representation. For $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ we have the so-called discrete series of representations. To each eigenvalue $c = k(k-1)$ of the Casimir operator C correspond two possible values $k = \frac{1}{2} \pm \big[\frac{1}{4} + c\big]^{1/2}$. The state $|k,0\rangle$ is annihilated by the operator K_- and is therefore the state with the lowest weight. The states $|k,n\rangle$ (with fixed k) are orthonormal:

$$\langle k,m|k,n\rangle = \delta_{m,n} \qquad \sum_{n=0}^{\infty} |k,n\rangle\langle k,n| = I_k \qquad (m,n = 0,1,2,\ldots) \tag{5}$$

and they span an infinite-dimensional Hilbert space H_k.

$SU(1,1)$ coherent states can be defined as

$$|k,z\rangle = (1-|z|^2)^k \sum_{n=0}^{\infty} d(k,n) z^n |k,n\rangle \qquad d(k,n) = \left[\frac{\Gamma(n+2k)}{n!\Gamma(2k)}\right]^{1/2} \tag{6}$$

where $|z|<1$. An alternative equivalent definition is

$$\begin{aligned} |k,r,\theta,\lambda\rangle &= S(r,\theta,\lambda)|k,0\rangle = \exp\big(\mathrm{i}\lambda k\big)|k,z\rangle \\ S(r,\theta,\lambda) &= \exp\left\{-\tfrac{1}{2}r\mathrm{e}^{-\mathrm{i}\theta}K_+ + \tfrac{1}{2}r\mathrm{e}^{\mathrm{i}\theta}K_-\right\}\exp\big(\mathrm{i}\lambda K_z\big) \\ z &= -\tanh\big(\tfrac{1}{2}r\big)\exp\big(\mathrm{i}(\theta-\lambda)\big). \end{aligned} \tag{7}$$

The overlap of two of these states is

$$\langle k,z_1|k,z_2\rangle = (1-|z_1|^2)^k(1-|z_2|^2)^k(1-z_1^* z_2)^{-2k}. \tag{8}$$

For $k>\frac{1}{2}$ we can write the following resolution of the identity [2–7] in terms of a surface integral of the states $|k,z\rangle$ over the unit disc D ($|z|<1$):

$$\frac{2k-1}{\pi}\int_D \mathrm{d}\mu(z)\ |k,z\rangle\langle k,z| = I_k \qquad \mathrm{d}\mu(z) = \frac{\mathrm{d}^2 z}{\big(1-|z|^2\big)^2}. \tag{9}$$

For later purposes we briefly prove this relation. We substitute $\mathrm{d}^2 z = \frac{1}{2}\mathrm{d}t\,\mathrm{d}\phi$, (where $z=\sqrt{t}\exp(\mathrm{i}\phi)$ and $0<t<1$) in (9) and integrate over the angle ϕ, to obtain

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k-1)}\left[\int_0^1 t^n(1-t)^{2k-2}\,\mathrm{d}t\right]|n,k\rangle\langle n,k| = I_k. \tag{10}$$

It is known that for $\lambda>0$ and $\mu>0$

$$\int_0^1 \mathrm{d}x\ x^{\lambda-1}(1-x)^{\mu-1} = \mathrm{B}(\lambda,\mu) \tag{11}$$

where

$$\mathrm{B}(\lambda,\mu) \equiv \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)} \tag{12}$$

is the Euler Beta function. In this way equation (9) is proved for $k>\frac{1}{2}$.

For $k<\frac{1}{2}$ the integral of (10) diverges (for t near to 1). In section 6 we shall regularize the divergent integral and extend this to the case $k<\frac{1}{2}$.

We now define the 'complementary states' which will be used in the next section for the formation of resolutions of the identity that involve $SU(1,1)$-coherent states on contours around the origin. They are defined as

$$\begin{aligned} |k,z;\mathrm{com}\rangle &= \big\{K(k,|z|)\big\}^{-1}\sum_{n=0}^{\infty}\big\{d(k,n)(z^*)^{n+1}\big\}^{-1}|k,n\rangle \\ \langle k,z;\mathrm{com}| &= \big\{K(k,|z|)\big\}^{-1}\sum_{n=0}^{\infty}\big\{d(k,n)z^{n+1}\big\}^{-1}\langle k,n| \\ K\big(k,|z|\big) &= \frac{\big\{F\big(1,1;2k;|z|^{-2}\big)\big\}^{1/2}}{|z|} \qquad |z|>1 \end{aligned} \tag{13}$$

where 'com' in the notation indicates complementary states, and F denotes the hypergeometric functions. Note that the normalization factor $K(|z|)$ converges only outside the unit disc. So the $SU(1,1)$-coherent states are defined inside the unit disc and their complementary states outside the unit disc.

It can easily be seen that

$$\langle k, z_2; \mathrm{com}|k, z_1\rangle = \left\{K(k, |z_2|)\right\}^{-1}\left(1 - |z_1|^2\right)^k \frac{1}{z_2 - z_1}. \tag{14}$$

Note that since z_1 is inside the unit disc and z_2 is outside the unit disc, one has $|z_1| < |z_2|$. This is needed because the overlap of these two states is expressed as a sum which converges to the right-hand side only if $|z_1| < |z_2|$.

We can also prove the following relation for the overlap of two complementary states:

$$\langle k, z_1; \mathrm{com}|k, z_2; \mathrm{com}\rangle = \frac{|z_1 z_2|}{z_1 z_2^*} \frac{F\left(1, 1; 2k; (z_1 z_2^*)^{-1}\right)}{\left[F(1, 1; 2k; |z_1|^{-2})F(1, 1; 2k; |z_2|^{-2})\right]^{1/2}}. \tag{15}$$

In many formulae below it will be convenient to use the 'analytic states'

$$|k, z\rangle_{\mathrm{anal}} = \sum_{n=0}^{\infty} d(k, n) z^n |k, n\rangle \tag{16}$$

$$|k, z; \mathrm{com}\rangle_{\mathrm{anal}} = \sum_{n=0}^{\infty} \left\{d(k, n)(z^*)^{n+1}\right\}^{-1} |k, n\rangle. \tag{17}$$

We call them analytic because their overlap with other states $|g\rangle$ is an analytical function of z, in an appropriate region (equation (16) when used as $\langle g|k, z\rangle_{\mathrm{anal}}$ and equation (17) when used as ${}_{\mathrm{anal}}\langle k, z; \mathrm{com}|g\rangle$). These states are not normalized to 1, but they belong to the Hilbert space H_k if their normalization is finite.

We now prove the following resolution of the identity in terms of the complementary states outside the unit disc $D^*(|z| > 1)$ and for $0 < k < \frac{1}{2}$:

$$\frac{\sin 2k\pi}{\pi^2} \int_{D^*} \mathrm{d}^2 z \, \frac{|k, z; \mathrm{com}\rangle_{\mathrm{anal}\,\mathrm{anal}}\langle k, z; \mathrm{com}|}{(|z|^2 - 1)^{2k}} = I_k. \tag{18}$$

In order to prove this, we substitute $\mathrm{d}^2 z = -(1/2t^2)\mathrm{d}t\,\mathrm{d}\phi$, (where $z = \exp(\mathrm{i}\phi)/\sqrt{t}$ and $0 < t < 1$) in (18) and use the following integral representation of the Beta function:

$$\mathrm{B}(n + 2k, 1 - 2k) = \int_0^1 \mathrm{d}t \, \frac{t^{n+2k-1}}{(1 - t)^{2k}}. \tag{19}$$

This integral representation of $\mathrm{B}(n + 2k, 1 - 2k)$ holds only for $0 < k < \frac{1}{2}$, whereas for $\frac{1}{2} < k$ this integral diverges. Finally we use the following well-known identity for products of Gamma functions

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin x\pi} \tag{20}$$

and prove equation (18). In section 6 we shall consider the case $\frac{1}{2} < k$ and regularize the divergent integral.

3. Weak resolutions of the identity in terms of contour integrals of *SU*(1, 1) coherent states

In order to write a resolution of the identity in terms of contour integrals that involve both the $SU(1,1)$ coherent states (defined inside the unit disc), and their complementary states (defined outside the unit disc), we need to be able to use both of them in a small ring

$$S\left(1 - \epsilon_2, 1 + \epsilon_1\right) = \left\{1 - \epsilon_2 < |z| < 1 + \epsilon_1\right\} \tag{21}$$

in the neighbourhood of the unit circle $|z| = 1$. In order to do this, we first point out that if $|f\rangle$ is an arbitrary normalized state in the Hilbert space H_k,

$$|f\rangle = \sum_{n=0}^{\infty} f_n |k, n\rangle \qquad \sum_{n=0}^{\infty} |f_n|^2 = 1 \tag{22}$$

then the function

$$F(z) \equiv \left(1 - |z|^2\right)^{-k} \langle f|k, z\rangle = \sum_{n=0}^{\infty} d(k, n) f_n^* z^n \tag{23}$$

defined initially within the unit disc, may be extended analytically if the expansion on the right-hand side converges on a larger disc. For example, in the special case of states $|f\rangle$ which are superpositions of a finite number of states, $|k, n\rangle$, the sum appearing in (23) is defined in the whole complex plane.

We call $H_k(1+\epsilon_1; A)$ (where $\epsilon_1 \geqslant 0$) the subset of the Hilbert space H_k which contains all the states $|f\rangle$ for which the sum (23) converges in the disc $|z| < 1 + \epsilon_1$. The A in the notation indicates the sum of (23). It is clear that if $\epsilon_1 < \epsilon_1'$ then the $H_k(1+\epsilon_1'; A)$ is a subset of $H_k(1 + \epsilon_1; A)$. An example of states that belong in the $H_k(1 + \epsilon_1; A)$ are the $SU(1, 1)$ coherent states $|k, z_0\rangle$ with $|z_0| < (1+\epsilon_1)^{-1}$. This example shows that for a *fixed non-zero* value of ϵ_1, the space $H_k(1+\epsilon_1; A)$ is *not* dense in H_k. Indeed all $SU(1, 1)$ coherent states $|k, z_0\rangle$ with $1 > |z_0| > (1 + \epsilon_1)^{-1}$ do not belong to the closure of the space $H_k(1 + \epsilon_1; A)$. However, in the limit $\epsilon_1 \to 0$ the $H_k(1 + \epsilon_1; A)$ becomes the Hilbert space H_k.

In an analogous way if $|g\rangle$ is an arbitrary normalized state

$$|g\rangle = \sum_{n=0}^{\infty} g_n |k, n\rangle \qquad \sum_{n=0}^{\infty} |g_n|^2 = 1 \tag{24}$$

then the quantity

$$G(z) \equiv K(k, |z|)\langle k, z, \mathrm{com}|g\rangle = \sum_{n=0}^{\infty} \frac{g_n}{d(k, n) z^{n+1}} \tag{25}$$

converges at least in the region $|z| > 1$, and possibly in a larger region.

We call $H_k(1 - \epsilon_2; B)$ (where $\epsilon_2 \geqslant 0$) the subset of the Hilbert space H_k which contains all the states $|g\rangle$ for which (25) converges in the region $|z| > 1 - \epsilon_2$. The B in the notation indicates the sum of (25). It is clear that if $\epsilon_2' > \epsilon_2$ then the $H_k(1 - \epsilon_2'; B)$ is a subset of $H_k(1 - \epsilon_2; B)$. An example of states that belong in the $H_k(1 - \epsilon_2; B)$ are the complementary states $|k, z_0; \mathrm{com}\rangle$ with $|z_0| > (1 - \epsilon_2)^{-1}$. This example shows that for a *fixed non-zero* value of ϵ_2, the space $H_k(1 - \epsilon_2; B)$ is *not* dense in H_k. Indeed all the complementary states $|k, z_0; \mathrm{com}\rangle$ with $1 < |z_0| < (1 - \epsilon_2)^{-1}$ do not belong to the closure of the space $H_k(1 - \epsilon_2; B)$. However, in the limit $\epsilon_2 \to 0$ the $H_k(1 - \epsilon_2; B)$ becomes the Hilbert space H_k.

Now let $|f\rangle$ be an arbitrary state in $H_k(1 + \epsilon_1; A)$ and $|g\rangle$ an arbitrary state in $H_k(1 - \epsilon_2; B)$. If C is an anticlockwise contour around the origin within the ring $S(1 - \epsilon_1, 1 + \epsilon_2)$ we can prove that

$$\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left(1 - |z|^2\right)^{-k} K(k, |z|)\langle f|k, z\rangle\langle k, z; \mathrm{com}|g\rangle = \langle f|g\rangle. \tag{26}$$

Indeed, substitution of (23), (25) into (26) gives the relation

$$\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left(\sum_{n=0}^{\infty} d(k, n) f_n^* z^n\right)\left(\sum_{m=0}^{\infty} \frac{g_m}{d(k, m) z^{m+1}}\right) = \langle f|g\rangle \tag{27}$$

which is easily proved using the formula

$$\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\frac{z^n}{z^{m+1}} = \delta_{nm}. \tag{28}$$

Therefore for bra states in $H_k(1+\epsilon_1; A)$, ket states in $H_k(1-\epsilon_2; B)$ and contours C in the ring $S(1-\epsilon_2, 1+\epsilon_1)$ we can rewrite (26) in the form of a resolution of the identity I_k:

$$\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\left(1-|z|^2\right)^{-k} K(k,|z|)|k,z\rangle\langle k,z;\mathrm{com}| = I_k. \tag{29}$$

We call this a weak resolution of the identity because there is some restriction on the bra and ket states that we can apply on the left and right of this identity. It is clear of course that ϵ_2 and ϵ_1 can be arbitrarily small and in fact one of them can even be zero. All that we need for the proof of (26) and (27) is a ring $S(1-\epsilon_2, 1+\epsilon_1)$ of finite width. Equation (29) can also be written in terms of the analytic states of (16), (17) as

$$\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,|k,z\rangle_{\mathrm{anal}\ \mathrm{anal}}\langle k,z;\mathrm{com}| = I_k. \tag{30}$$

The above formalism clearly defines the conditions under which our weak resolution of the identity is valid.

4. Expansion of an arbitrary state in terms of *SU*(1, 1) coherent states on a contour

Using equation (29) we can express an arbitrary state $|g\rangle$ within $H_k(1-\epsilon_2; B)$ as a superposition of $SU(1,1)$ coherent states on a contour C in the ring $S(1-\epsilon_2, 1)$:

$$|g\rangle = \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,g(z)|k,z\rangle \tag{31}$$

with

$$\begin{aligned} g(z) &= \left(1-|z|^2\right)^{-k} K(k,|z|)\langle k,z;\mathrm{com}|g\rangle \\ &= \left(1-|z|^2\right)^{-k} G(z) \\ &= \left(1-|z|^2\right)^{-k} \sum_{n=0}^{\infty} \frac{g_n}{d(k,n)z^{n+1}}. \end{aligned} \tag{32}$$

We consider several examples. The first one is the states $|k,n\rangle$ for which equations (31) and (32) easily give

$$|k,n\rangle = \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,\left(1-|z|^2\right)^{-k} \frac{1}{d(k,n)z^{n+1}}|k,z\rangle. \tag{33}$$

Note that for the states $|k,n\rangle$ the sum of (23) converges everywhere in the complex plane (apart from the origin). Therefore C can be any anticlockwise contour in the unit disc around the origin. We can easily check that this result is correct, if we substitute $|k,z\rangle$ from (6) in (33).

Another example is the $SU(1,1)$ coherent states $|k,z_0\rangle$ ($|z_0| < 1$). Equations (31) and (32) in conjuction with (14) give

$$|k,z_0\rangle = \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}}\left(\frac{1-|z_0|^2}{1-|z|^2}\right)^k \frac{1}{z_0-z}|k,z\rangle. \tag{34}$$

In this case the sum of (25) converges only if $|z_0| < |z|$. This leads to the conclusion that the contour C should be in the ring $S(1-|z_0|, 1)$. Note that in this case the contour C will enclose the pole at z_0. It is easy to check that in this case the right-hand side of (34) is indeed equal to $|k, z_0\rangle$.

Another example is the complementary states $|k, z_0; \mathrm{com}\rangle$ ($|z_0| > 1$). Using equation (15) in conjunction with (31) and (32) we find that

$$|k, z_0; \mathrm{com}\rangle = \oint_C \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{|z_0|}{zz_0^*} \frac{F\left(1, 1; 2k; (zz_0^*)^{-1}\right)}{[F(1, 1; 2k; |z_0|^{-2})]^{1/2}} \left(1 - |z|^2\right)^{-k} |k, z\rangle. \qquad (35)$$

In this case the sum of (25) converges only if $|zz_0| > 1$. This leads to the conclusion that the contour C should be in the ring $S(|z_0|^{-1}, 1)$.

5. Application of the formalism to squeezed states

In this section we apply the above formalism to squeezed states. We consider the following well-known realizations of $SU(1, 1)$ generators with single boson operators:

$$K_- \equiv \tfrac{1}{2}a^2 \qquad K_+ \equiv \tfrac{1}{2}a^{\dagger 2} \qquad K_0 \equiv \tfrac{1}{4}\left(aa^\dagger + a^\dagger a\right) \qquad C = -\tfrac{3}{16}I. \qquad (36)$$

In this case $c = -\frac{3}{16}$ corresponding to $k = \frac{1}{4}$ and $k = \frac{3}{4}$. The correspondence between the states $|k, n\rangle$ of (5) and the usual harmonic oscillator number eigenstates $|n\rangle$ is as follows:

$$\left|\tfrac{1}{4}, n\right\rangle \equiv |2n\rangle \qquad \left|\tfrac{3}{4}, n\right\rangle \equiv |2n + 1\rangle. \qquad (37)$$

The Hilbert space $H_{1/4}$ is isomorphic to the even Fock subspace (i.e. the Fock subspace spanned by the even number eigenstates); the $H_{3/4}$ is isomorphic to the odd Fock subspace. Therefore the Hilbert space of the harmonic oscillator is isomorphic to the direct sum $H_{1/4} + H_{3/4}$.

The $SU(1, 1)$ operators of (7) are in this case the squeezing operators. $SU(1, 1)$-coherent states are defined as

$$\begin{aligned} \left|\tfrac{1}{4}, z\right\rangle &= \left(1 - |z|^2\right)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} z^n \left|\tfrac{1}{4}, n\right\rangle, \\ \left|\tfrac{3}{4}, z\right\rangle &= \left(1 - |z|^2\right)^{3/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} z^n \left|\tfrac{3}{4}, n\right\rangle. \end{aligned} \qquad (38)$$

The states $|\frac{1}{4}, z\rangle$ are squeezed vacua; and the states $|\frac{3}{4}, z\rangle$ are the number eigenstate $|1\rangle$ squeezed.

The complementary states are defined for $k = \frac{1}{4}$ as

$$\begin{aligned} \left|\tfrac{1}{4}, z; \mathrm{com}\right\rangle &= \left\{K\left(\tfrac{1}{4}, |z|\right)\right\}^{-1} \sum_{n=0}^{\infty} \frac{2^n n!}{[(2n)!]^{1/2} (z^*)^{n+1}} |2n\rangle \\ \left\langle\tfrac{1}{4}, z; \mathrm{com}\right| &= \left\{K\left(\tfrac{1}{4}, |z|\right)\right\}^{-1} \sum_{n=0}^{\infty} \frac{2^n n!}{[(2n)!]^{1/2} z^{n+1}} \langle 2n| \\ K\left(\tfrac{1}{4}, |z|\right) &= \frac{\left\{F\left(1, 1; \frac{1}{2}; |z|^{-2}\right)\right\}^{1/2}}{|z|} \\ &= \frac{1}{(|z|^2 - 1)^{3/4}} \left\{\arcsin\left(\frac{1}{|z|}\right) + \left(|z|^2 - 1\right)^{1/2}\right\}^{1/2} \qquad |z| > 1 \end{aligned} \qquad (39)$$

and for $k = \frac{3}{4}$ as

$$\begin{aligned}
\left|\tfrac{3}{4}, z; \mathrm{com}\right\rangle &= \left\{K\left(\tfrac{3}{4}, |z|\right)\right\}^{-1} \sum_{n=0}^{\infty} \frac{2^n n!}{[(2n+1)!]^{1/2}(z^*)^{n+1}} |2n+1\rangle \\
\left\langle\tfrac{3}{4}, z; \mathrm{com}\right| &= \left\{K\left(\tfrac{3}{4}, |z|\right)\right\}^{-1} \sum_{n=0}^{\infty} \frac{2^n n!}{[(2n+1)!]^{1/2} z^{n+1}} \langle 2n+1| \\
K\left(\tfrac{3}{4}, |z|\right) &= \frac{\left\{F\left(1, 1; \frac{3}{2}; |z|^{-2}\right)\right\}^{1/2}}{|z|} \\
&= \frac{1}{(|z|^2-1)^{1/4}} \left[\arcsin\left(\frac{1}{|z|}\right)\right]^{1/2} \qquad |z| > 1.
\end{aligned} \tag{40}$$

Let $|f\rangle$ be a normalized state in the harmonic oscillator Hilbert space:

$$|f\rangle = \sum_{n=0}^{\infty} f_n |k, n\rangle \qquad \sum_{n=0}^{\infty} |f_n|^2 = 1. \tag{41}$$

According to our terminology introduced earlier, we will say that this state belongs in the direct sum $H_{1/4}(1+\epsilon_1; A) + H_{3/4}(1+\epsilon_1'; A)$ if the sums

$$\begin{aligned}
\left(1-|z|^2\right)^{-1/4} \left\langle f \middle| \tfrac{1}{4}, z\right\rangle &= \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} [f_{2n}]^* z^n \\
\left(1-|z|^2\right)^{-3/4} \left\langle f \middle| \tfrac{3}{4}, z\right\rangle &= \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} [f_{2n+1}]^* z^n
\end{aligned} \tag{42}$$

converge for $|z| < 1+\epsilon_1$ and $|z| < 1+\epsilon_1'$, correspondingly. We will also say that $|f\rangle$ belongs in the direct sum $H_{1/4}(1-\epsilon_2; B) + H_{3/4}(1-\epsilon_2'; B)$ if the sums

$$\begin{aligned}
K\left(\frac{1}{4}, |z|\right) \left\langle \tfrac{1}{4}, z, \mathrm{com} \middle| f\right\rangle &= \sum_{n=0}^{\infty} \frac{2^n n! f_{2n}}{[(2n)!]^{1/2} z^{n+1}} \\
K\left(\frac{3}{4}, |z|\right) \left\langle \tfrac{3}{4}, z, \mathrm{com} \middle| f\right\rangle &= \sum_{n=0}^{\infty} \frac{2^n n! f_{2n+1}}{[(2n+1)!]^{1/2} z^{n+1}}
\end{aligned} \tag{43}$$

converge for $|z| > 1-\epsilon_1$ and $|z| > 1-\epsilon_1'$, correspondingly.

Now let $|f\rangle$ be a state in $H_{1/4}(1+\epsilon_1; A) + H_{3/4}(1+\epsilon_1'; A)$ and $|g\rangle$ be a state in $H_{1/4}(1-\epsilon_2; B) + H_{3/4}(1-\epsilon_2'; B)$. If C_0 is an anticlockwise contour around the origin within the ring $S(1-\epsilon_1, 1+\epsilon_2)$ and C_1 is an anticlockwise contour around the origin within the ring $S(1-\epsilon_1', 1+\epsilon_2')$ we can prove that

$$\begin{aligned}
&\oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left(1-|z|^2\right)^{-1/4} K\left(\tfrac{1}{4}, |z|\right) \left\langle f \middle| \tfrac{1}{4}, z\right\rangle \left\langle \tfrac{1}{4}, z; \mathrm{com} \middle| g\right\rangle \\
&\qquad + \oint_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left(1-|z|^2\right)^{-\frac{3}{4}} K\left(\tfrac{3}{4}, |z|\right) \left\langle f \middle| \tfrac{3}{4}, z\right\rangle \left\langle \tfrac{3}{4}, z; \mathrm{com} \middle| g\right\rangle = \langle f|g\rangle.
\end{aligned} \tag{44}$$

Therefore for bra states in $H_{1/4}(1+\epsilon_1; A) + H_{3/4}(1+\epsilon_1'; A)$, ket states in $H_{1/4}(1-\epsilon_2; B) + H_{3/4}(1-\epsilon_2'; B)$ and contours C_0 in $S(1-\epsilon_1, 1+\epsilon_2)$ and C_1 in $S(1-\epsilon_1', 1+\epsilon_2')$ we can rewrite (44) in the form of a resolution of the identity

$$\oint_{C_0} \frac{dz}{2\pi i} \left(1 - |z|^2\right)^{-1/4} K\left(\tfrac{1}{4}, |z|\right) \left|\tfrac{1}{4}, z\right\rangle \left\langle \tfrac{1}{4}, z; \text{com}\right|$$

$$+ \oint_{C_1} \frac{dz}{2\pi i} \left(1 - |z|^2\right)^{-3/4} K\left(\tfrac{3}{4}, |z|\right) \left|\tfrac{3}{4}, z\right\rangle \left\langle \tfrac{3}{4}, z; \text{com}\right| = I_k. \tag{45}$$

These results show how the general formalism developed earlier can be used in the context of squeezed states in quantum optics.

6. Extended resolutions of the identity: regularization of the divergent integrals

The resolution of the identity (9) is valid only for $k > \frac{1}{2}$ (because for $k < \frac{1}{2}$ the integral diverges); and the resolution of the identity (18) is valid only for $k < \frac{1}{2}$ (because for $k > \frac{1}{2}$ the integral diverges). In this section we regularize these divergent integrals and consider the special case $k = \frac{1}{2}$ by a limiting procedure.

This involves the $SU(1, 1)$ coherent states and also the complementary states on the circle $|z| = 1$. Although these states are not normalizable, in this paper we have carefully defined how these states can be used by taking their overlap with states in appropriate spaces. For the $SU(1, 1)$ coherent states with $|z| = 1$ the appropriate space is $H_k(1 + \epsilon; A)$ where ϵ is any positive number; and for the complementary states with $|z| = 1$ the appropriate space is $H_k(1 - \epsilon; B)$ where ϵ is any positive number.

As an application of this formalism we now use these states on the circle $|z| = 1$ to regularize the divergent integral (9) for $k < \frac{1}{2}$, and the divergent integral (18) for $k > \frac{1}{2}$.

6.1. Weak resolutions of the identity in terms of surface integrals of the SU(1,1) coherent states with $0 < k < \frac{1}{2}$

For $k < \frac{1}{2}$ the integral of (10) diverges (for t near to 1). However it can be regularized as explained in [13, page 66] in the following way. Consider

$$\int_0^1 dt\ \frac{t^n}{(1-t)^{2-2k}} = \int_0^1 dt\ \frac{1}{(1-t)^{2-2k}} - \int_0^1 dt\ \frac{1-t^n}{(1-t)^{2-2k}}. \tag{46}$$

The regularization occurs when we formally use the relation

$$\int_0^1 dt\ \frac{1}{(1-t)^{2-2k}} = \mathrm{B}(1, 2k-1) = \frac{1}{2k-1} \tag{47}$$

for the first integral, for $k < \frac{1}{2}$ [13]. This formula (for $k < \frac{1}{2}$) is to be understood as a regularization relation rather than a standard integral. The second integral on the right-hand side of (46) is convergent (a factor $(1 - t)$ is in both the numerator and denominator and simplifies). In this way we obtain

$$\begin{aligned}\int_0^1 dt\ t^n (1-t)^{2k-2} &= \frac{1}{2k-1} - \int_0^1 dt\ \frac{1 + t + \cdots + t^{n-1}}{(1-t)^{1-2k}} \\ &= \frac{1}{2k-1} - \sum_{l=0}^{n-1} \mathrm{B}(l+1, 2k) = \mathrm{B}(n+1, 2k-1).\end{aligned} \tag{48}$$

The last equality in (48) can be proved inductively. We have checked numerically that the regularized integral in (48) is indeed equal to the Beta function.

Using equation (48) and the integral

$$\lim_{\varepsilon\to+0} \varepsilon \int_0^1 \mathrm{d}t \, \frac{t^n}{(1-t)^{1-\varepsilon}} = \lim_{\varepsilon\to 0} \frac{\varepsilon\Gamma(\varepsilon)\Gamma(n+1)}{\Gamma(\varepsilon+n+1)} = 1 \tag{49}$$

we prove the following resolution of the identity for $0 < k < \frac{1}{2}$:

$$\lim_{\varepsilon\to+0} \varepsilon \int_D \frac{\mathrm{d}^2 z}{\pi} \, \frac{|k,z\rangle_{\text{anal anal}}\langle k,z|}{(1-|z|^2)^{1-\varepsilon}} + (1-2k) \int_D \frac{\mathrm{d}^2 z}{\pi} \, \frac{J(k,z)}{(1-|z|^2)^{2-2k}} = I_k. \tag{50}$$

where

$$J(k,z) = \big|k, z/|z|\big\rangle_{\text{anal anal}}\big\langle k, z/|z|\big| - |k,z\rangle_{\text{anal anal}}\langle k,z|. \tag{51}$$

The two integrals in (50) correspond to the two integrals in (46). The operator $J(k,z)$ corresponds to the numerator $(1-t^n)$ of the second integral on the right-hand side of (46). It contains $SU(1,1)$ coherent states on the circle $|z| = 1$ (which corespond to the 1 in the term $(1-t^n)$), and play a crucial role in the regularization of the integral. Therefore equation (50) is a weak resolution of the identity which can be used for states in a space $H_k(1+\epsilon; A)$ where ϵ is any positive number.

It is interesting to see what happens in the case $k = \frac{1}{2}$ because in this case the $SU(1,1)$ coherent states become phase states (for a discussion on phase states from this point of view see for example [7, 10]; for a general discussion on phase operators and phase states see [14]). The resolution of the identity for $k = \frac{1}{2}$ can be obtained from (9) by taking the limit $k \to \frac{1}{2}$ from above. Another way to obtain the resolution of the identity for $k = \frac{1}{2}$ is to take the limit $k \to \frac{1}{2}$ from below in (50). In both cases we obtain the relation [10]

$$\begin{aligned} I_{1/2} &= \lim_{\varepsilon\to+0} \varepsilon \int_D \frac{\mathrm{d}^2 z}{\pi} \, \frac{\big|\frac{1}{2},z\big\rangle_{\text{anal anal}}\big\langle\frac{1}{2},z\big|}{(1-|z|^2)^{1-\varepsilon}} \\ &= \lim_{\varepsilon\to+0} \frac{1}{\log(1/\varepsilon)} \int_{|z|\leqslant 1-\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \, \frac{\big|\frac{1}{2},z\big\rangle_{\text{anal anal}}\big\langle\frac{1}{2},z\big|}{1-|z|^2}. \end{aligned} \tag{52}$$

A different resolution of the identity for the case $0 < k < \frac{1}{2}$ was studied in [15].

6.2. *Weak resolutions of the identity in terms of surface integrals of the complementary states with $k > \frac{1}{2}$*

The integral in (19) diverges when $k > \frac{1}{2}$ (for t near 1). We first consider the region $\frac{1}{2} < k < 1$ and regularize this integral by a modification of the procedure explained in [13] in the following way:

$$\begin{aligned} \int_0^1 \mathrm{d}t \, \frac{t^{n+2k-1}}{(1-t)^{2k}} &= \int_0^1 \mathrm{d}t \, \frac{t^{2k-1}}{(1-t)^{2k}} - \int_0^1 \mathrm{d}t \, \frac{t^{2k-1}(1-t^n)}{(1-t)^{2k}} \\ &= \mathrm{B}(2k, 1-2k) - \sum_{l=0}^{n-1} \int_0^1 \mathrm{d}t \, \frac{t^{l+2k-1}}{(1-t)^{2k-1}} \\ &= \mathrm{B}(2k, 1-2k) - \sum_{l=0}^{n-1} \mathrm{B}(l+2k, 2-2k) \\ &= \mathrm{B}(n+2k, 1-2k). \end{aligned} \tag{53}$$

The regularization occurs when we formally use the relation

$$\int_0^1 \mathrm{d}t \, \frac{t^{2k-1}}{(1-t)^{2k}} = \mathrm{B}(2k, 1-2k) = \frac{\pi}{\sin 2k\pi} \tag{54}$$

for the first integral in (53) which is divergent for $k > \frac{1}{2}$. The last equality in (53) can be proved inductively.

Using equations (53) we prove the following resolution of the identity in case of $\frac{1}{2} < k < 1$:

$$\lim_{\varepsilon\to+0} \varepsilon \int_{D^*} \frac{\mathrm{d}^2 z}{\pi} \, \frac{|k, z; \mathrm{com}\rangle_{\mathrm{anal\ anal}}\langle k, z; \mathrm{com}|}{(|z|^2-1)^{1-\varepsilon}} - \frac{\sin 2k\pi}{\pi} \int_{D^*} \frac{\mathrm{d}^2 z}{\pi} \, \frac{J(k, z; \mathrm{com})}{(|z|^2-1)^{2k}} = I_k \tag{55}$$

where

$$J(k, z; \mathrm{com}) = \big|k, z/|z|; \mathrm{com}\big\rangle_{\mathrm{anal\ anal}}\big\langle k, z/|z|; \mathrm{com}\big| - |k, z; \mathrm{com}\rangle_{\mathrm{anal\ anal}}\langle k, z; \mathrm{com}|. \tag{56}$$

Note that the operator $J(k, z; \mathrm{com})$ involves complementary states on the circle $|z| = 1$, which play a crucial role in the regularization of the integral. Therefore (55) is a weak resolution of the identity which can be used for states in a space $H_k(1-\epsilon; B)$ where ϵ is any positive number.

The resolution of the identity (55) fails for $1 < k$ because the second integral becomes divergent. As explained in [13] in these cases one has to make further regularizations of the divergent integrals for each of the regions $1 < k < \frac{3}{2}$, $\frac{3}{2} < k < 2$, ... separately. The cases $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ can be obtained by limiting procedures. We will not present this explicitly here.

7. Conclusion

We have expanded previous work that used resolutions of the identity in terms of contour integrals of coherent states [9–12] to the $SU(1,1)$ case. This required the definition of both $SU(1,1)$ coherent states and their complementary states in a ring $1-\epsilon_2 < |z| < 1+\epsilon_1$. In equation (20) we have defined coherent states in a disc $|z| < 1+\epsilon_1$. Their overlaps with states in the full Hilbert space do not necessarily exist; but their overlaps with states in a smaller space $H_k(1+\epsilon_1; A)$ do exist. Similarly in equation (22) we have defined complementary states for $|z| > 1-\epsilon_2$. Their overlaps with states in the full Hilbert space do not necessarily exist; but their overlaps with states in a smaller space $H_k(1-\epsilon_2; B)$ do exist.

With this construction we have shown that for bra states in $H_k(1+\epsilon_1; A)$ and ket states in $H_k(1-\epsilon_2; B)$ the resolution of the identity (19), is well defined. Using this we have expanded in (24) an arbitrary state in terms of $SU(1,1)$ coherent states. Several examples of this have been presented and the whole formalism has been applied to squeezed states in quantum optics.

The resolution of the identity (9) is known to be valid for $k > \frac{1}{2}$. As a byproduct of our formalism we have regularized the divergent integral and we have given an extension of this identity for $0 < k < \frac{1}{2}$ in (50). In a similar way, equation (18) is valid for $0 < k < \frac{1}{2}$; but we have regularized the divergent integral and we gave an extension of this for $k > \frac{1}{2}$ in (55).

The set of all coherent states is highly overcomplete and it is practically very useful to have resolutions of the identity in terms of smaller sets of coherent states. In this paper we have studied resolutions of the identity in terms of contour integrals of $SU(1,1)$ coherent states. The work can be applied to many areas, in particular in quantum optics (where a scheme for the experimental production of states similar to our complementary states has recently been proposed in [16]).

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